# Generalized Functions Exercise 1

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1. Let $f \in L^1_{loc}(R)$. We need to show that $\xi_f$ defined by $\xi_f(g) = \int_{-\infty}^{\infty} f(x)g(x)dx$ for any $g \in C_c^\infty(R)$ is adistribution. Linearity is clear by the elementary properties of the integral, thus it remains to be shown that for $g_n, g \in C_c^\infty(\mathbb{R})$ such that $g_n$ converge to $g$ (in the sense defined in class) we have:

$$\xi_f(g_n) = \int_{-\infty}^{\infty} f(x)g_n(x)dx \to \xi_f(g) = \int_{-\infty}^{\infty} f(x)g(x)dx$$

This Follows from the dominated convergence theorem: indeed, we have pointwise convergence, and for n sufficiently large $\mid f(x)g_n(x) \mid \leq \mid f(x)(\mid g(x) \mid + 1_{[-M,M]}) \mid$(where M is large enough that $[-M, M]$ contatins the support of $g_n, g)$ ,which is integrable.

2. Let $U_1, U_2 \subseteq \mathbb{R}$ be open sets and let $g \in C_c^\infty(U_1 \cup U_2)$ . Let $K \subseteq U_1 \cup U_2$ be the (compact) support of $g$. We claim that one can find compact sets$K_1 \subseteq U_1$, $K_2 \subseteq U_2$ such that $K \subseteq K_1 \cup K_2$. Indeed, $U_1, U_2$ are open so for every point in $K$ one can find a neighborhood whose (compact) closure is contained in either $U_1$or $U_2$ (by taking any open neighborhood and shrinking it). By compatness one can then take a finite subcover. Now take $K_1$ to be the union of the closures of the neighborhoods contained in $U_1$ (which is compact as a union of finitely many compact subsets of $R$), and similarly for $K_2$. Now we can construct cut off functions for $K_1$and$K_2$, i.e functions $h_1, h_2$ in $C_c^\infty(U_1)$, $C_c^\infty(U_2)$ that

are identically 1 in $K_1, K_2$ respectively (This construction was shown in the tirgul). We claim that $g = gh_1 + gh_2(h_1 - 1)$ (note that $gh_1 \in C_c^\infty(U_1)$ and $gh_2(1 - h_1) \in C_c^\infty(U_2)$) . Indeed, for points outside of $K$ we have $0 = 0 + 0$. For a point in $K$, if it's in $K_1$we have $gh_1 + gh_2(1 - h_1) = g + 0 = g$, and if it's in $K_2$we get $gh_1 + gh_2(1 - h_1) = gh_1 + g(1 - h_1) = g$ . So the desired equality holds throughout $U_1 \cup U_2$ and we are done.

3. We define a linear mapping from the space of equivalence classes of cauchy sequences of weakly convergent smooth functions with compact support to the space of distributions as follows: $D([f_n])(g) = lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx$ (this limit exists by the definition of a cauchy sequence and is well defined by the definition of equivalence). Clearly $D$ is a linear operator. The continuity of $D$ follows from the following, more general claim: the limit of a sequence of weakly convergent distributions is itself a distribution (a sequence of distributions $D_n$ is called weakly convergent if $D_n(f)$ converges for any $f \in C_c^\infty(\mathbb{R})$). This in turn follows from the Banach-Steinhaus theorem for Frechet spaces: a functional on $C_c^\infty(\mathbb{R})$ is continuous iff its rectriction to $C_c^\infty(K)$ is continuous for every compact subset $K \subseteq \mathbb{R}$ (by definition of sequential continuity, which implies continuity for functionals). This space is a Frechet Space and thus by Banach-Steinhaus a pointwise limit of continuous functionals on it is continuous.

This mapping is canonical in the sense that it takes a constant sequence $f_n \equiv f$ to the distribution corresponding to $f$. It remains to be shown that this map is an isomorphism: injectivity is clear, because by definition $D([f_n])$ is the zero distribution iff $[f_n] = 0$. We claim that given some distribution $D$ one can find a sequence of compactly supported smooth functions that approximates it:

First, for any $n$ let $g_n \in C_c^\infty(\mathbb{R})$ be a function that is identically 1 on $[-n, n]$. Then clearly for any distribution $D$, $f \in C_c^\infty(\mathbb{R})$, $g_n D(f) = D(g_n f) = D(f)$ for $n$ sufficiently large.

Now let $f_n \in C_c^\infty(\mathbb{R})$ be an approximation to the identity (i.e for all $n$, $f_n \geq 0$, $\int_{-\infty}^{\infty} f_n(x)dx = 1$ and $supp(f_n)$ shrinks to $\{0\}$). Note that for any $g \in C_c^\infty(R)$, $g * f_n$ tends to $g$ (strongly in $C_c^\infty(\mathbb{R})$). Indeed, $g, g * f_n$ are all supported in some compact set because $supp(f_n)$ shrinks to$\{0\}$ and $supp(g * f_n) \subseteq supp(g) + supp(f_n)$. Uniform convergence follows easily from the uniform continuity of $g$, and uniform convergence of the derivatives then follows from the identity $(g * f_n)^{'} = g^{'} * f_n$.

We claim that for any ditribution $D$, $f_n * D \to D$ (note that $f_n * D$ is smooth because its derivative equals $f_n^{'} * D$). This is true because for any $g \in C_c^\infty(\mathbb{R})$ we have

$$D(g(-t)) = g*D(0) = lim_{n\to\infty}(g*f_n)*D(0) = lim_{n\to\infty}g*(f_n*D)(0) = lim_{n\to\infty}(f_n*D)(g(-t))$$

where we use the associativity of convolution. Finally, we can exhibit a sequence of functions in $C_c^\infty(\mathbb{R})$ converging (weakly) to D: the sequence $g_n(f_n * D)$. This is true because of a combination of the above arguments: for any $h \in C_c^\infty(R)$, $g_n(f_n * D)(h) = f_n * D(h)$ for $n$ sufficiently large, and this tends to $D(h)$.

4. (a) We need to show that $supp(a\xi_1 + b\xi_2) \subseteq supp(\xi_1) \cup supp(\xi_2)$.If $U$ is open and $\xi_1, \xi_2$ both vanish on $U$, then clearly so does $a\xi_1 + b\xi_2$. Therefore we have that $(supp(\xi_1) \cup supp(\xi_2))^c = supp(\xi_1)^c \cap supp(\xi_2)^c$, which is the union of all such $U$, is cotained in $(supp(a\xi_1 + b\xi_2))^c$, and we are done.

(b) We need to show that $supp(\xi) \cap int(supp(\xi)) \subseteq supp(\xi^{'}) \subseteq supp(\xi)$. The second inclusion is obvious- if $\xi$ vanishes in some open set then clearly the same holds for $\xi^{'}$and we are done. To prove the first inclusion we use the following lemma: let $\xi$ be a distribution such that $\xi^{'}$vanishes on $U = (a, b) \subseteq \mathbb{R}$. Then $\xi$ is constant on $(a, b)$, i.e there is some constant $c$ such that $\xi(f) = c\int_a^b f(x)dx$ for any $f \in C_c^\infty(U)$.

Proof: Note that for $g \in C_c^\infty(U)$, $g$ is the derivative of a test function iff $\int_a^b g(x)dx = 0$ (indeed, this is equivalent to $G(x) = \int_a^x g(x)dx$ being compactly supported). For any such $g$ we have $\xi(g) = -\xi^{'}(G) = 0$. Now fix some $h \in C_c^\infty(U)$ with $\int_a^b h(x)dx = 1$. Then for any $f \in C_c^\infty(U)$ we have $\int_a^b (f(x) - h(x)\int_a^b f(t)dt)dx = 0$, so $\xi(f) = \xi(h)\int_a^b f(x)dx$, and $c = \xi(h)$ is our required constant.

Now suppose $x \in supp(\xi) \cap int(supp(\xi))$ but also $x \notin supp(\xi^{'})$. Then by definition $\xi^{'}$vanishes in some neighboorhood of $x$,and thus by our lemma $\xi$ is constant there. Now this constant must be non zero, because otherwise $x \notin supp(\xi)$. But this implies that in this neighboorhood any point is in $supp(\xi)$ (because, again, $\xi$ is a non zero constant there), so $x \in int(supp(\xi))$,contradicion.

6. We need to show that the convolution of distributions with compact support is associative. So let $S, T, U$ be distributions with compact support. To simplify the notation, we write the variable with respect to which the distribution is acting in sub-script, so for instance we write $f(x) = (U * g)(x)$ as $U_t(g(x-t))$. Now take some function $h \in C_c^\infty(\mathbb{R})$. Note that we have for any two distributions with compact support

$$(S * T)(h) = S_t(T_x(h(x+t)))$$

So $(S * T) * U(h) = (S * T)_t(U_x(h(x+t))$. Denote $f(t) = U_x(h(x+t))$.

Then $(S*T)*U(h) = (S*T)(f) = S_z(T_u(f(z+u)) = S_z(T_u(U_x(h(x+z+u))))$. Now we apply the identity $(S * T)(h) = S_t(T_x(h(x+t)))$ twice to obtain

$$S_z(T_u(U_x(h(x+z+u)))) = S_z((T * U)_t(h(t+z))) = (S * (T * U))(h)$$

so we have $(S * T) * U = S * (T * U)$ and we are done.

7. We need to show that for $K \subseteq \mathbb{R}$ compact, a functional $\xi : C_c^\infty(K) \to \mathbb{R}$

is continuous iff there exists some $k \geq 0$ and $c > 0$ such that for all $f \in C_c^\infty(K)$ we have

$$| \xi(f) | \leq c \| f \|_{C^k}$$

one direction is clear- if $\xi$ is bounded in the above sense then it is clearly continuous at 0, and therefore by linearity everywhere. Conversely, suppose $\xi$ is continuous. Assume that $\xi$ isn't bounded: this implies the existence of a sequence $f_n \in C_c^\infty(K)$ such that for all $n$

$$| \xi(f_n) | > n \| f_n \|_{C^n}$$

by rescaling we can can assume $\xi(f_n) = 1$ for all $n$. This implies that

$$1/n > \| f_n \|_{C^n} = sup_{x \in K} \sum_{i=1}^{i=n} | f_n^{(i)}(x) | \geq sup_{x \in k} | f_n^{(j)}(x) |$$

for any $j \leq n$. Fixing $j$ and letting $n$ tend to infinity, we get that $f_n$ and all their derivatives tend uniformly to 0, and furthermore we know that their supports are all contained in the compact set $K$. So $f_n$ tend to 0 (in the strong sense). But $\xi(f_n) = 1$ for all $n$, contradicting the continuity of $\xi$.

8. (a) Note that away from 0, $G$ is some solution of the homogenous differential equation $A(G) = 0$. Thus to specify $G$ it suffices to describe its behaviour at 0. Write $A = a_n d^n + a_{n-1} d^{n-1} + ... a_o d^0$ (where $a_n \neq 0$). We claim that if $G$ is a solution to Green's equation it satisfies the following: $G, ..., G^{n-2}$ are continuous at 0, and $G^{n-1}$ is discontinuous there with $lim_{\varepsilon \to 0^+} G^{n-1}(\varepsilon) - G^{n-1}(-\varepsilon) = \frac{1}{a_n}$. The continuity condition follows from the fact that if $G^{(i)}$had a jump discontinuity at 0 for $i \leq n-2$ , we would get that near 0 $G^{(i+1)} \propto \delta_0$, and thus $G^{(n)} \propto \delta_0^{(k)}$, for some $k \geq 2$. Indeed, if a function $f$ has a jump discontinuity at

$x$ but is smooth elswhere, we have for any $g \in C_c^\infty(\mathbb{R})$:

$$f^{'}(g) = -\int_{-\infty}^{\infty} f(y)g^{'}(y)dx = lim_{\varepsilon\to 0} - \int_{|y-x|>\varepsilon} f(y)g^{'}(y) = lim_{\varepsilon\to 0}(-\left[f(y)g(y)\right]_{x+\varepsilon}^{\infty} - \left[f(y)g(y)\right]_{-\infty}^{x-\varepsilon} +$$

$$\int_{|y-x|>\varepsilon} f'(y)g(y)dy = g(x)(lim_{\varepsilon\to 0} f(x+\varepsilon) - f(x-\varepsilon)) + \int_{-\infty}^{\infty} f^{'}(y)g(y)dy$$

But the other side of the equation contains only $\delta_0$, and $\delta_0, ..., \delta_0^{(k)}$ are independent. Thus $G, ..., G^{n-2}$ are continuous. To determine the size of the discontinuity of $G^{n-1}$, we take $\varepsilon > 0$ and integrate the equation, getting:

$$1 = \int_{-\epsilon}^{\epsilon} \delta = \int_{-\epsilon}^{\epsilon} a_n \frac{d}{dx} G^{(n-1)}(x)dx + \int_{-\varepsilon}^{\varepsilon} a_{n-1} \frac{d}{dx} G^{(n-2)}(x)dx + ... + \int_{-\varepsilon}^{\varepsilon} a_0 G(x)dx$$

Now we take $\varepsilon \to 0$, and observe that from the continuity of $G, .., G^{n-2}$, all the terms except $lim_{\varepsilon\to 0} \int_{-\epsilon}^{\varepsilon} a_n \frac{d}{dx} G^{(n-1)}(x)dx$ vanish. So we are left with

$$1 = lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} a_n \frac{d}{dx} G^{(n-1)}(x)dx = a_n lim_{\varepsilon\to 0}(G^{(n-1)}(\varepsilon) - G^{(n-1)}(-\varepsilon))$$

and we get the size of the jump. Conversely, suppose $A(G) = 0$ away from 0, and $G$ satisfies the conditions above.

(b) Denote by $G_A(x, y)$ the solution of $A(G)(y) = \delta(y - x)$. For some $g \in C_c^\infty(\mathbb{R})$, set $A_{G_A}(g)(y) = \int_{-\infty}^{\infty} G_A(x, y)g(x)dx$. We need to prove the identity $A(A_{G_A}(g)(y)) = g(y)$. This follows from the properties of the green function and the $\delta$ function:

$$A(A_{G_A}(g)(y)) = A\int_{-\infty}^{\infty} G_A(x, y)g(x)dx = \int_{-\infty}^{\infty} A(G_A(x, y))g(x)dx = \int_{-\infty}^{\infty} \delta(y-x)g(x)dx = g(y)$$